4.2 ODE review

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Def. 4.1 A *differential equation* of order n is an equation

$$f\left(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \dots, \frac{d^nx}{dt^n}, t\right) = 0,$$ where x is a function of t.

$$f(x, \dot{x}, \ddot{x}, \dots, \overset{(n)}{x}, t)$$

Def. If an ODE does not depend explicitly on t, i.e.

$$f(x, \dot{x}, \dots, \overset{(n)}{x}, t) \equiv f(x, \dot{x}, \dots, \overset{(n)}{x}),$$

then it is *autonomous*. Otherwise, *nonautonomous*.

Def. An ODE that can be rewritten as

$$\frac{d^nx}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \dots + a_{n-1}(t)\frac{dx}{dt} + a_n(t)\,x = g(t).$$

is *linear*. Otherwise, *nonlinear*.

Def. If given a linear ODE of the form above, $g(t) \equiv 0$, then it is *homogeneous*. Otherwise, *nonhomogeneous*.

Def. A *first-order system* can be written as

$$\frac{dX}{dt} = F(X(t), t)$$

where

$$X(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad F = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}, \quad f_i \equiv f_i(x_1(t), x_2(t), \dots, x_n(t), t)$$

Def. A first-order system is *autonomous* if $F(X(t), t) \equiv F(X(t))$, i.e. does not depend explicitly on t. *Nonautonomous* otherwise.

Def. A first-order system is *linear* if

$$\frac{dX}{dt}(t) = A(t)X(t) + G(t)$$

where

$$A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix}, \quad G(t) = \begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

Otherwise, nonlinear.

Def. A linear first-order system is **homogeneous** if $G(t) \equiv \vec{0}$.
Otherwise, **nonhomogeneous**.

Thm 4.1 (i) Let $\overset{(n)}{x} + a_1(t)\overset{(n-1)}{x} + \cdots + a_{n-1}(t)\dot{x} + a_n(t)x = g(t)$,

$x(t_0) = x_0$, $\dot{x}(t_0) = x_1$, ..., $\overset{(n-1)}{x}(t_0) = x_{n-1}$ initial conditions.

Then if $\{a_i\}_{i=1,2,\dots,n-1}$, and g are continuous on an open interval I containing t_0, then $\exists$ a unique solution $x(t)$ on I.

(ii) Let $\dot{X}(t) = A(t)X(t) + G(t)$, $X(t_0) = X_0$ initial cond.

Then if $A(t)$, $G(t)$ are continuous on an open interval I containing t_0, then $\exists$ unique soln $X(t)$ on I.

Ex. 4.1 $\dfrac{d^2x}{dt^2} + \dfrac{3}{t}\dfrac{dx}{dt} + \dfrac{x}{t^2} = 0$, $x(1) = 0$, $\dot{x}(1) = 1$.

All coefficients are cont. on $(0, \infty)$, so $\exists$ unique soln.

We can find a general soln $x_{gen}(t) = c_1 t^{-1} + c_2 t^{-1} \ln t$

$$\dot{x}_{gen}(t) = -c_1 t^{-2} + c_2 t^{-2} - c_2 t^{-2} \ln t$$

Solve the IVP $\quad x(1) = 0 = c_1$

$$\dot{x}(1) = 1 = -c_1 + c_2 \Rightarrow c_2 = 1$$

$$\Rightarrow x(t) = t^{-1} \ln t.$$